Chair of Applied Mechanics
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Model Reduction to Invariant Manifolds

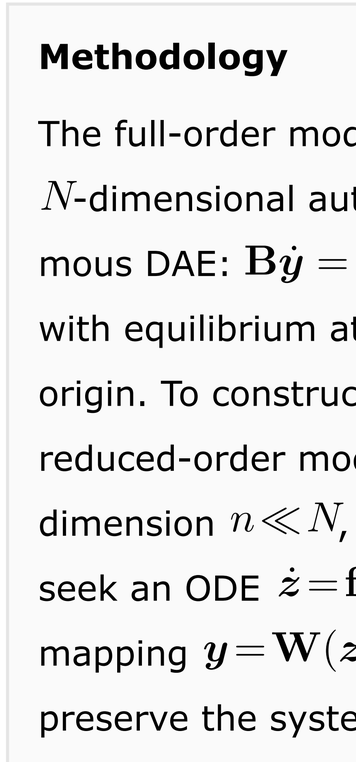
Overview on the Direct Parameterisation of Invariant Manifolds

Methodology

The full-order model is an N-dimensional autonomous DAE: $\mathbf{B}\dot{\boldsymbol{y}} = \mathbf{A}(\boldsymbol{y})$, with equilibrium at the origin. To construct a reduced-order model of dimension $n \ll N$, we seek an ODE $\dot{\boldsymbol{z}} = \mathbf{f}(\boldsymbol{z})$ and mapping $\boldsymbol{y} = \mathbf{W}(\boldsymbol{z})$ that preserve the system's essential dynamics near the equilibrium. Directly solving the resulting functional equation, $\mathbf{B}\,\nabla\mathbf{W}\cdot\mathbf{f} = \mathbf{A}\circ\mathbf{W}$, is still intractable. Hence, we assume power series expansions for $\mathbf{A}$, $\mathbf{W}$ and $\mathbf{f}$. So, the coefficients can be determined recursively for each monomial $\boldsymbol{z}^{\mathbf{P}} = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}$ via the corresponding homological equation.

Adaptable to:

- Mechanical systems
- Smooth forcing
- Algebraic constraints
- Varying parameters
- Retain variables of the full model in the ROM



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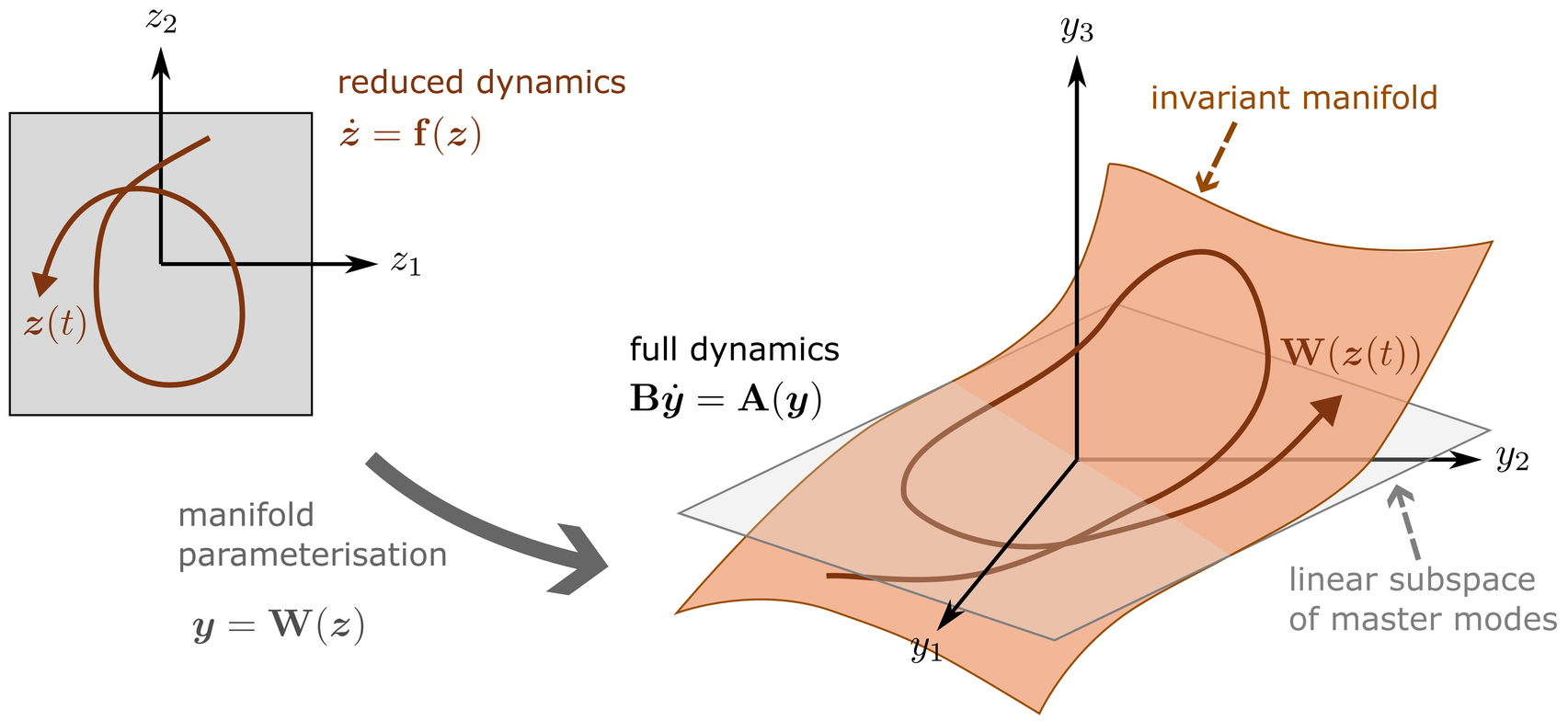
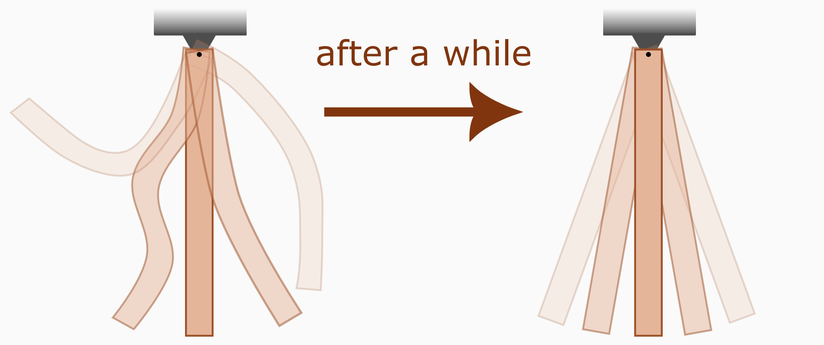
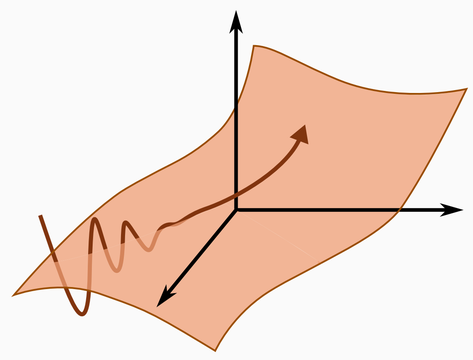


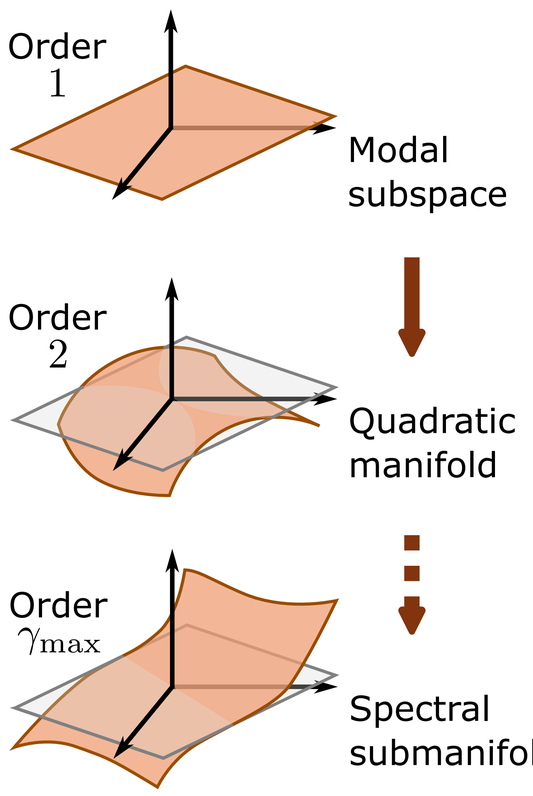
Slow and Attracting Invariant Manifold



The system quickly tends to states inside the invariant manifold that has slow dynamics.



Inserting the ROM in the full dynamics yields the invariance equation: $\mathbf{B}\,\nabla\mathbf{W}\cdot\mathbf{f} = \mathbf{A}\circ\mathbf{W}$.



$$\mathbf{A}(\boldsymbol{y}) = \sum \mathbf{A}_{\boldsymbol{\alpha}}\, \boldsymbol{y}^{\boldsymbol{\alpha}}, \quad \mathbf{W}(\boldsymbol{z}) = \sum \mathbf{W}_{\boldsymbol{\sigma}}\, \boldsymbol{z}^{\boldsymbol{\sigma}}, \quad \mathbf{f}(\boldsymbol{z}) = \sum \mathbf{f}_{\boldsymbol{\nu}}\, \boldsymbol{z}^{\boldsymbol{\nu}}$$

$$\boldsymbol{y}^{\boldsymbol{\alpha}} = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_N^{\alpha_N}$$

Find master eigenmodes, i.e. order 1 monomials.

$$\mathbf{B}\mathbf{W}_1\mathbf{f}_1 = \mathbf{A}_1\mathbf{W}_1$$
$$\rightsquigarrow [\lambda_k\mathbf{B} - \mathbf{A}_1]\mathbf{W}_{\mathbf{e}_k} = \mathbf{0}$$

$\gamma = 2$

Solve all monomials $\mathbf{p}$ of order $|\mathbf{p}| = \gamma$.

$$\mathbf{B}\mathbf{W}_1\mathbf{f}_{\boldsymbol{p}} + [(\boldsymbol{\lambda}\cdot\boldsymbol{p})\mathbf{B} - \mathbf{A}_1]\mathbf{W}_{\boldsymbol{p}} = \mathrm{RHS}(\mathbf{A}_{\leq|\boldsymbol{p}|}, \mathbf{W}_{<|\boldsymbol{p}|}, \mathbf{f}_{<|\boldsymbol{p}|})$$

$\gamma \leftarrow \gamma + 1$

Reference

Vizzaccaro, A., Gobat, G., Frangi, A. et al. Direct parametrisation of invariant manifolds for non-autonomous forced systems including superharmonic resonances. Nonlinear Dyn 112, 6255–6290 (2024).