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ROTATIONAL MOTIONS OF SATELLITES IN A RADIAL GRAVITY FIELD

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Summary

It is currently desirable to have a knowledge of the stability and transient behavior of arbitrary shapes of satellites with arbitrary orientations in a radial gravity field. Such knowledge, unfortunately, can be obtained only for certain restricted types of motion. Several types of motion have been treated under restrictions listed in the introduction. The general equations of motion for a two body problem with gravity as the only acting force are treated. The approach differs from other authors in that the attracting potential is not used. A treatment which in a certain sense summarizes and discusses the results of other authors is given. Approximate equations for the motion of satellites of arbitrary shapes are presented for the assumption that the size of the satellite is small relative to the earth. For symmetrical satellites, the stable configurations and the transient behavior of the unstable ones are determined and presented in graphical form. Calculations show that the transient behavior is nearly independent of the length of the satellite.

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1. Introductory Remarks

Motions of celestial bodies having nearly spherical ellipsoids of inertia are usually described with sufficient accuracy by the Kepler equations. In mechanics of satellites however, motions of rod-shaped and disc-shaped bodies are also of interest. Therefore, more general equations for a two-body problem must be considered.

The differential equations of motion for the mutual gravitational attraction of two finite rigid bodies contain the equations for forces and torques. These equations describe position and angular velocity of the satellite but are complicated by the coupling of vibrational and orbital motions. It is necessary to combine these equations with geometrical equations of the satellite to describe its orientation in space.

Certain aspects of this problem have been treated by several authors in recent years. Roberson made general calculations for the gravitational torques acting about the center of mass of a satellite (1) and also derived general equations for rotational motions of satellites with moving parts inside (2). The same problem was treated by Lurye (3); however, neither author gave a solution for a practical system of coordinates. In a series of publications, Duboschin (4, 5, 6) treated several aspects of rotational motion. After deriving general equations for the motion of n rigid bodies with gravitational attraction, he restricts his consideration to a two-body problem with a homogeneous sphere (earth) and a rod-shaped satellite. For this case, three special solutions -- the so-called "regular motions" -- were calculated. Up to now, these solutions seem to be the only exact solutions for the equations in question.

Considerations concerning these special solutions are discussed in detail in Chapter 3. Duboschin (6) also gave approximate results for satellites with symmetrical ellipsoids of inertia. However, because he uses a system of sophisticated variables, physical interpretation of his results is difficult. Parts of his discussion are therefore completed and extended in Chapter 3.

There are also other publications related to the problem in question. Davis (7) calculated possible vibrations of a simplified satellite consisting of two masses separated by a fixed distance (dumbbell-shaped). However, generalization of these results to satellites with arbitrary shapes does not meet the requirements of an exact mathematical analysis. The simplifications involved require that the results be interpreted with extreme caution. Stocker and Vachino (8) treated a special case of motion of a dumbbell-shaped satellite in an elliptical orbit. Their results show that vibrations of the dumbbell axis in the orbital plane can take place with varying amplitude and frequency. Angular motions of bodies of arbitrary shape were calculated by Suddath (9). But, because he neglects the coupling between orbital and rotational motion and assumes special kinds of acting torques not corresponding to gravitational torques, his considerations do not apply to the type of motion of interest in this investigation. The same applies to an investigation conducted by Cole, Epstrand, and O'Neill (10). They neglect from the start the gravitational torques and solve the classical equations for the rotation of a rigid body for several cases of special external torques.

At the time that this investigation was nearing completion, the author became aware of several other reports related to this subject. They are listed as numbers 11 to 14 in the references. The essence of each report is briefly stated. Klemperer (11) indicated an exact solution for the vibrations in the plane of the orbit of a dumbbell-shaped satellite. His solution is analogous to the familiar motion of a pendulum and can be expressed by elliptical integrals. Beletskiy (12) calculated the stability of the positions of equilibrium relative to the earth for non-spinning satellites of arbitrary shape. His calculations are valid for the more general elliptical orbit, but they are restricted to satellites which are small relative to the earth. Thomson (13) treated the special case of a symmetrical satellite in a circular orbit oriented such that its axis of symmetry is perpendicular to the plane of the orbit. His calculations of the amount of spin necessary to stabilize the satellite appear in figure 5 of this report in a completed form. Thomson only calculated the region of vibrational instability, whereas fig. 5 contains the region of asymptotic instability as well. A very comprehensive study of the investigations which have been conducted in the field of stability of satellites is presented in a Ph.D. thesis by DeBra (14). His personal contribution concerns the effects which ellipticity of the orbit, oblateness of the earth, and inner damping have on stability of the satellite.

The following considerations shall contribute to an understanding of the complicated motions of satellites in orbit. Because the practical problem is an extremely difficult one, this investigation is restricted by the following assumptions:

1. Forces and torques arising from atmospheric resistance, magnetic fields, and solar-pressure shall be neglected.
2. Classical Newtonian mechanics and the Newtonian gravitational law shall be valid.
3. Only a two body problem is treated; effects of other celestial bodies are neglected.

4. The earth is a homogeneous sphere (or is composed of homogeneous concentric spherical shells).
5. The center of mass of the whole system (earth and satellite) is stationary or moves without acceleration.
6. The satellite is a rigid body.

In Chapter 4, it will furthermore be assumed that the dimensions of the satellite are small compared with the radius of the earth. Special assumptions concerning the shape of the satellite are made in Chapter 3 where a rod-shaped satellite is treated and in Chapter 5 where ellipsoids of inertia are assumed to be rotational ellipsoids.

This restricted problem is much simpler than the real motion of satellites. But, considering our present state of knowledge, the problem can only be formulated clearly with these restrictions.

Extreme caution must be observed in calculating the behavior of satellites due to the difference between celestial and terrestrial mechanics. Effects negligible in terrestrial mechanics are frequently quite appreciable in celestial mechanics. An example is the concept of center of gravity which cannot be used in this treatment. A center of gravity simply does not exist because the directions of the resulting gravity forces do not pass through a fixed "center" in the body if the orientation of the body in the radial gravity field varies. This investigation reveals that, even for very small satellites, these effects cannot be neglected if long periodic motions are of interest.

The basic equations are given not in potential form but in vector notation, using the equations of equilibrium of forces and torques. Also, the geometrical description of body orientation shall be presented in a somewhat different way from most of the other authors. The well-known Euler-angles

ϕ, Ψ, Θ shall not be used because they lose their uniqueness in the special case $\Theta = 0$. It is therefore more convenient to take either the directional cosines or another set of angles. This makes it easier to interpret the physical significance of theoretical results.

2. The Basic Equations

The basic equations of the satellite are established from the system of position vectors shown in Fig. 1. Vector $\bar{r}$ defines the distance between center of mass O_E of the earth and a particle in the satellite, $\bar{r}_E$ the distance between center of mass O of the whole system and center of mass O_E of the earth, $\bar{r}_S$ the distance between center of mass O of the whole system and center of mass O_S of the satellite, and $\bar{r}'$ the distance between center of mass O_S of the satellite and a particle in the satellite. These vectors have components in each of two different coordinate systems, one inertial and the other body fixed. Inertial coordinates are ξ, η, ζ with origin at O and body fixed x,y,z with origin at O_S.

By representing masses of the earth and satellite by m_E and m_S respectively, the following equations are readily established:

$$m_E \bar{r}_E + m_S \bar{r}_S = 0$$

and
$$\bar{r} = \bar{r}_S - \bar{r}_E + \bar{r}' = \frac{m}{m_E} \bar{r}_S + \bar{r}' = \bar{r}_A + \bar{r}'$$

where
$$m = m_E + m_S \; ; \quad r_A = r_S - r_E = \frac{m}{m_E} \bar{r}_S$$

The vector $\bar{r}_A$ is the displacement between points O_E and O_S.

The mass particle dm is attracted by gravitational forces and by inner forces $d\bar{F}_i$. Using Newton's third law, there follows

$$\frac{d^2}{dt^2}(\bar{r}_s + \bar{r}')\,dm = (\ddot{\bar{r}}_s + \ddot{\bar{r}}')\,dm = d\bar{F} \tag{2}$$
$$= -\gamma \frac{m_E\,dm}{r^3} \bar{r} + d\bar{F}_i$$

The product of gravitational constant γ and mass of the earth can be expressed as

$$\gamma m_E = g_o R^2 \tag{3}$$

where R is radius of the earth and g_o is surface acceleration of the earth. When equation (2) is integrated over the satellite, the following terms vanish

$$\int d\overline{F}_i = 0 \quad \text{AND} \quad \int \ddot{\overline{r}}' dm = 0$$

The first vanishes because internal forces cancel and the second because, by definition of center of mass,

$$\int \overline{r}' dm = 0$$

Equation (2) then becomes

$$m_s \ddot{\overline{r}}_s + \gamma m_E \int \frac{\overline{r}\, dm}{r^3} = 0 \tag{4}$$

Equilibrium of torques yields another equation. If $\overline{H}_o$ and $\overline{T}_o$ are vectors representing, respectively, angular momentum of, and torque acting on, the satellite (both originating from the origin O of the inertial system), then by the law of angular momentum

$$\dot{\overline{H}}_o = \overline{T}_o \tag{5}$$

where

$$\overline{H}_o = \int \left[(\overline{r}_s + \overline{r}') \times \frac{d}{dt} (\overline{r}_s + \overline{r}') \right] dm$$

Angular momentum about the body fixed origin O_s is

$$\overline{H}_s = \int (\overline{r}' \times \dot{\overline{r}}) \, dm$$

Because

$$\int \overline{r}' dm = 0 \quad \text{AND} \quad \int \dot{\overline{r}}' dm = 0$$

$\overline{H}_o$ reduces to

$$\overline{H}_o = m_s (\overline{r}_s \times \dot{\overline{r}}_s) + \overline{H}_s$$

In addition, validity of the following relation

$$\int (\bar{r}_s + \bar{r}') \times d\bar{F}_i = 0$$

permits the equation for torque to be written

$$\bar{T}_o = -\gamma m_E \int (\bar{r}_s + \bar{r}') \times \frac{\bar{r}}{r^3} \, dm$$

Using equation (4) and the equation for torque, equation (5) become

$$\dot{\bar{H}}_s = -\gamma m_E \int \frac{\bar{r}' \times \bar{r}}{r^3} \, dm \qquad (6)$$

This is the equation of angular momentum for the moving point O_s as a reference point. Transforming equation (6) to the body fixed system gives

$$\frac{d\bar{H}_s}{dt} = \frac{d'\bar{H}_s}{dt} - \bar{H}_s \times \bar{\omega} = -\gamma m_E \int \frac{\bar{r}' \times \bar{r}}{r^3} \, dm \qquad (7)$$

where $\bar{\omega}$ is the body rotation vector and the prime indicates derivation with respect to time in the moving coordinate system. Using the relation

$$\dot{\bar{r}}' = \bar{\omega} \times \bar{r}'$$

where $\bar{r}'$ is a body fixed vector, the angular momentum of the satellite becomes

$$\bar{H}_s = \int [\bar{r}' \times (\bar{\omega} \times \bar{r}')] \, dm \qquad (8)$$

Equations (4) and (7) are equations for the vectors $\bar{r}_s$ and $\bar{\omega}$. Simultaneous solutions to these equations will reveal the position of the center of mass of the satellite and its angular velocity. This does not reveal orientation of the satellite however. A relation for orientation can be obtained by considering the corresponding kinematical equations. If $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$ are the unit vectors for the x, y, z coordinates of the body fixed system, the vector $\bar{r}'$ can be expressed as

$$\bar{r}' = \bar{e}_x x + \bar{e}_y y + e_z z$$

Also
$$\dot{\bar{e}}_x = \bar{\omega} \times \bar{e}_x$$
$$\dot{\bar{e}}_y = \bar{\omega} \times \bar{e}_y \qquad (10)$$
$$\dot{\bar{e}}_z = \bar{\omega} \times \bar{e}_z$$

and
$$\left.\begin{array}{ll} \bar{e}_i \cdot \bar{e}_j = 0 & i \neq j \\ \bar{e}_i \cdot e_j = 1 & i = j \end{array}\right\} i, j = x, y, z$$

These relations are sufficient to calculate the nine directional cosines (the components of $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$ in the inertial system ξ, η, ζ) giving orientation of the satellite in space.

For certain calculations it is more convenient to use a system of three independent angles instead of the nine interrelated directional cosines. The frequently used Euler-angles ϕ , Ψ, Θ are sufficient to describe the orientation of the satellite in space, but they have the disadvantage of losing their uniqueness in the case $\Theta = 0$. This case, however, can be of interest for certain applications. Therefore a set of similar angles ϕ , Ψ, δ with $\delta = \frac{\pi}{2} - \Theta$ as shown in Figure 2 shall be used. For $|\delta| < \frac{\pi}{2}$ these angles give a unique representation of the satellite's orientation.

The matrix of transformations between the body-fixed x, y, z system and the space-fixed ξ, η, ζ system is given by the following:

	ξ	η	ζ
x	$\cos\phi\cos\psi - \sin\phi\sin\psi\sin\delta$	$\cos\phi\sin\psi + \sin\phi\cos\psi\sin\delta$	$\sin\phi\cos\delta$
y	$-\sin\phi\cos\psi - \cos\phi\sin\psi\sin\delta$	$-\sin\phi\sin\psi + \cos\phi\cos\psi\sin\delta$	$\cos\phi\cos\delta$
z	$\sin\psi\cos\delta$	$-\cos\psi\cos\delta$	$\sin\delta$

(11)

The components of rotation in the body-fixed system can be expressed by the angles ϕ , Ψ, δ as follows:

$$\omega_x = \dot{\psi} \sin \phi \cos \delta - \dot{\delta} \cos \phi$$
$$\omega_y = \dot{\psi} \cos \phi \cos \delta + \dot{\delta} \sin \phi$$
$$\omega_z = \dot{\psi} \sin \delta + \dot{\phi} \tag{12}$$

Equations (4), (7), and (10) represent a system of twelfth order which may be reduced by use of energy and angular momentum relations; however, this does not simplify the system enough for general explicit solutions to result. The principal difficulty encountered in solving these equations is due to the dependence of the integrals in (4) and (7) upon both ellipsoid of inertia and orientation of the satellite. This implies coupling between the orbital and rotational motions; separate treatment is not possible for a more exact analysis. The special case of a rod-shaped satellite will be considered in the next chapter. Approximate solutions involving a minimum of simplification of the problem will be considered later.

3. Special Solutions for Rod-Shaped Satellites

Three "regular motions" for the special two-body problem of a spherical earth and a rod-shaped satellite were obtained by Duboschin. In each of the three cases, the center of mass of the satellite moved with constant orbital speed on a circular orbit around the earth. These cases differed only in orientation of the satellite relative to the earth. Orientation in each case was

Case I. longitudinal axis perpendicular to the plane of the orbit.

Case II. longitudinal axis tangential to the orbit itself.

Case III. longitudinal axis pointing through the center of the earth.

These three modes of motion can be derived as particular solutions of equations (4) and (7). To show this, let the z-axis of the body fixed coordinate system be collinear with the longitudinal axis of the satellite. Then $\bar{r}' = \bar{e}_z z$ and $dm = \mu dz$ where μ is the mass per unit length of the rod. If the length of the satellite is 2L, integration over the satellite must be from $z = -L$ to $z = +L$.

For Case I, the following relations arise:

$$r_A = \text{const.}\ ; \quad \bar{r}_A \bar{r}' = 0\ ; \quad r^2 = r_A^2 + r'^2 = r_A^2 + z^2 \qquad (13)$$

and, from equation (4),

$$\int \frac{\bar{r}\, dm}{r^3} = \mu\, \bar{r}_A \int_{-L}^{L} \frac{dz}{[r_A^2 + z^2]^{\frac{3}{2}}} + \mu\, \bar{e}_z \int_{-L}^{L} \frac{z\, dz}{[r_A^2 + z^2]^{\frac{3}{2}}}$$

The first term to the right of the equal sign vanishes because it is the integral of an odd function over an area symmetrical to $z = 0$. The equation then becomes, using $m_s = 2\mu L$,

$$\int \frac{\bar{r}\, dm}{r^3} = \mu\, \bar{r}_A \left[\frac{z}{r_A^2 \sqrt{r_A^2 + z^2}} \right]_{-L}^{L} = \frac{m_s\, \bar{r}_A}{r_A^2 \sqrt{r_A^2 + L^2}} \qquad (14)$$

From equation (7), the integral

$$\int \frac{\overline{r}' \times \overline{r}}{r^3}\, dm = \int \frac{\overline{r}' \times \overline{r}_A}{r^3}\, dm = \overline{e}_Z\, \mu\, r_A \int_{-L}^{L} \frac{z\, dz}{[r_A^2 + Z^2]^{\frac{3}{2}}} \tag{15}$$

also vanishes because the integrand is odd. Consequently, the basic equations (4) and (7) have the form

$$\ddot{\overline{r}}_s + \frac{\gamma m}{r_A^2 \sqrt{r_A^2 + L^2}}\, \overline{r}_s = \ddot{\overline{r}}_s + \Omega_I^2\, \overline{r}_s = 0 \tag{16}$$

$$\frac{d' H_s}{dt} - H_s \times \overline{\omega} = 0 \tag{17}$$

Equation (16) gives as a special solution a vector of constant length rotating with constant angular velocity Ω in a plane determined by the initial velocity and center of the earth. The time for one orbit becomes

$$T_I = \frac{2\pi}{\Omega_I} = 2\pi \sqrt{\frac{r_A^2 \sqrt{r_A^2 + L^2}}{\gamma m}} \tag{18}$$

For discussion it is convenient to introduce r_A = R+h where h is altitude of the satellite above the earth's surface. Equation (18) can then be written as

$$T_I = T_o \sqrt{\frac{m}{m_E}} \sqrt{\left(1 + \frac{h}{R}\right)^3} \sqrt[4]{1 + \left(\frac{L}{R+h}\right)^2} \tag{19}$$

where $$T_o = 2\pi \sqrt{\frac{R}{g_o}}$$

is the well-known Schuler-period of 84.3 minutes which was detected by Schuler during his efforts to avoid erroneous indications of navigational instruments when moving on the earth's surface. This period may be thought of as a general earth bound time constant which at the same time is the shortest possible time of revolution for an earth satellite.

Equation (17) is always satisfied in Case I because $\overline{H}_S = 0$. The components of $\overline{H}_S$ are $A\omega_x$, $B\omega_y$, $C\omega_z$ where A, B, C are the principal moments of inertia. For a rod-shaped satellite, A = B and C = 0. Also, $\overline{\omega} = \overline{e}_Z \omega_Z$ and $\omega_x = \omega_y = 0$. Thus, the rod may rotate with any arbitrary but constant speed about its z-axis (the longitudinal axis).

In Case II, the rod orbits with its longitudinal axis tangential to the orbit and normal to vector $\overline{r}_A$. Therefore, relations (13), (14), (15), (16), and (17) are also valid for this case and $\Omega_{II} = \Omega_I$ and $T_{II} = T_I$. Using $\overline{\omega} = (\omega_x, \omega_y, \omega_z)$ and $\overline{H}_s = (A\omega_x, A\omega_y, 0)$, the three scalar equations for equation (17) are

$$\dot{\omega}_x - \omega_y \omega_z = 0$$
$$\dot{\omega}_y + \omega_x \omega_z = 0$$
$$\dot{\omega}_z = 0$$

Thus, the rod in this case may also rotate around its longitudinal axis with an arbitrary angular velocity ω_z. A possible solution to this set of scalar equations is

$$\omega_z = \text{CONST.}$$
$$\omega_x = \Omega_{II} \cos \omega_z t$$
$$\omega_y = \Omega_{II} \sin \omega_z t$$

This result is compatible with the geometrical conditions of Case II.

In Case III, the rod orbits with its longitudinal axis pointing through the center of the earth and collinear with vector $\overline{r}_A$. Consequently, the following relations arise:

$$\overline{r}' = \overline{e}_z r' = \overline{e}_z Z$$
$$\overline{r}_A = \overline{e}_z r_A$$
$$\overline{r} = \overline{r}_A + \overline{r}' = \overline{e}_z (r_A + Z)$$
$$\overline{r}' \times \overline{r} = 0$$

The integral in equation (7) vanishes and the integral in (4) becomes

$$\int \frac{\overline{r}\, dm}{r^3} = \mu\, \overline{e}_z \int_{-L}^{L} \frac{dZ}{(r_A + Z)^2} = \overline{e}_z \left[\frac{m_s}{r_A^2 - L^2}\right]$$

Therefore, the basic equations are

$$\ddot{\overline{r}}_s + \frac{\gamma\, m_E}{r_s (r_A^2 - L^2)} \overline{r}_s = \ddot{\overline{r}}_s + \Omega_{III}^2 \overline{r}_s = 0 \tag{20}$$

$$\frac{d' \overline{H}_s}{dt} - \overline{H}_s \times \overline{\omega} = 0 \tag{21}$$

A possible solution of the first equation is a circular orbit with constant speed of revolution. The time for one complete revolution is

$$T_{III} = \frac{2\pi}{\Omega_{III}} = T_o\sqrt{\frac{m_E}{m}}\sqrt{\left(1+\frac{h}{R}\right)^3}\sqrt{1-\left(\frac{L}{R+h}\right)^2} \tag{22}$$

Consideration of the three scalar equations reveals that rotation around the longitudinal axis has no influence on the motion as was found in Cases I and II.

In summarizing, circular orbits for the center of mass of the satellite are possible for all three cases. The time for one complete revolution differs from the Schuler-period T_o = 84.3 minutes by correction factors relating to:

(a) the height $\sqrt{\left(1+\frac{h}{R}\right)^3}$

(b) the mass relation $\sqrt{\frac{m_E}{m}}$

(c) the orientation $\sqrt[4]{1+\left(\frac{L}{R+h}\right)^2}$ Case I and II

$\sqrt{1-\left(\frac{L}{R+h}\right)^2}$ Case III

Factor (a) is the same as for the ideal Kepler motion. Factor (b) differs from unity by only 10^{-22} for a one ton satellite and hence can be neglected. Factor (c) differs from unity by about 10^{-6} for a satellite thirty feet long. Considering the desired accuracy of present day space endeavors, this effect can be of interest. A rod-shaped satellite for Cases I and II orbits slightly slower and for Case III slightly faster than a mass point on the same orbit.

Every orientation of the satellite for which the longitudinal axis is perpendicular to the vector $\overline{r}_A$ has the same equation of motion for Cases I and II. However, equation (17) has no solution which satisfies the geometrical conditions for the whole orbit -- except for the special Cases I and II. Because the principal moment of inertia C is zero, the vector of angular momentum $\overline{H}_S$ is always perpendicular to the axis of the rod. Furthermore, it is perpendicular to $\overline{r}_A$. For the satellite to have the same orientation relative to the earth, $\overline{H}_S$ must change its orientation in space. This is possible if and only if a corresponding torque is present. However, this is not the case in equation (17).

4. Approximations for Small Parameters $\epsilon = L/R$

Dimensions of artificial satellites up to the present have been small relative to the earth. If the size of the satellite is represented by L and radius of the earth by R, then the parameter $\epsilon = L/R$ will be small. When the integrands of the basic equations (4) and (7),

$$\frac{\bar{r}}{r^3} \quad \text{and} \quad \frac{\bar{r}' \times \bar{r}}{r^3} \quad \text{respectively,}$$

are developed in series of powers of ϵ, useful approximations of these equations result by neglecting higher powers of ϵ. The following developments are not specialized to shapes of satellites.

In developing the integrand of equation (4), the following expressions involving $\bar{r}$ are

$$\begin{aligned} \bar{r} &= \bar{r}_A + \bar{r}' \quad \text{WHERE} \quad r' \ll r_A \\ r^2 &= r_A^2 + r'^2 + 2(\bar{r}_A \bar{r}') \\ \frac{1}{r^3} &= r_A^{-3} - 3(\bar{r}_A \bar{r}') r_A^{-5} - \frac{3}{2} r'^2 r_A^{-5} + \frac{15}{2}(\bar{r}_A \bar{r}')^2 r_A^{-7} + \cdots \end{aligned} \tag{23}$$

Consequently,

$$\begin{aligned} \int \frac{\bar{r}\, dm}{r^3} &= \frac{m_s \bar{r}_A}{r_A^3} - \frac{3}{2} \bar{r}_A r_A^{-5} \int r'^2 dm \\ &+ \frac{15}{2} \bar{r}_A r_A^{-7} \int (\bar{r}_A \bar{r}')^2 dm - 3 r_A^{-5} \int \bar{r}' (\bar{r}_A \bar{r}') dm + \cdots \end{aligned} \tag{24}$$

Let

$$\bar{r}' = \bar{e}_x X + \bar{e}_y Y + \bar{e}_z Z$$

and

$$\bar{r}_A = r_A (\bar{e}_x \gamma_x + \bar{e}_y \gamma_y + \bar{e}_z \gamma_z)$$

where $\gamma_x, \gamma_y, \gamma_z$ are the direction cosines of vector $\bar{r}_A$ in the body fixed system. In addition, let it be assumed that the x, y, z system of coordinates corresponds to the system of principal axes of inertia with center of mass of the satellite at the origin O_S. Then the centrifugal moments vanish because

$$\int XY\, dm = \int YZ\, dm = \int ZX\, dm = 0 \tag{25}$$

From the principal moments of inertia

$$A = \int (y^2 + z^2)\, dm \,;\quad B = \int (z^2 + x^2)\, dm \,;\quad C = \int (x^2 + y^2)\, dm$$

and the polar moment of inertia

$$P = \int (x^2 + y^2 + z^2)\, dm = \frac{1}{2}(A + B + C)$$

it follows that $\int \bar{r}'\, dm = P$

$$\int (\bar{r}_A \bar{r}')^2\, dm = r_A^2 \left[P - (\gamma_x^2 A + \gamma_y^2 B + \gamma_z^2 C) \right] \qquad (26)$$

and

$$\int \bar{r}' (\bar{r}_A \bar{r}')\, dm = P \bar{r}_A - r_A (\bar{e}_x \gamma_x A + \bar{e}_y \gamma_y B + \bar{e}_z \gamma_z C)$$

Using equations (24) and (26), the basic equation (4) becomes

$$\ddot{\bar{r}}_s + \frac{\gamma m_E^3}{m^2} \frac{\bar{r}_s}{r_s^3} + \frac{3 \gamma m_E^5}{m_s m^4} \left[P - \frac{5}{2}(\gamma_x^2 A + \gamma_y^2 B + \gamma_z^2 C) \right] \frac{\bar{r}_s}{r_s^3}$$
$$+ \frac{3 \gamma m_E^5}{m_s m^4} \frac{1}{r_s^4} (\bar{e}_x \gamma_x A + \bar{e}_y \gamma_y B + \bar{e}_z \gamma_z C) + \cdots = 0 \qquad (27)$$

For the case of satellites with spherical ellipsoids of inertia, equation (27) becomes the well-known equation for Kepler motion. The additional terms in (27) can change both period and shape of an orbit. It is possible that the orbit no longer lies in a plane because the last term in (27) represents a force which generally does not go through the origin O. It can readily be seen that the two terms containing moments of inertia are smaller than the second term by a factor of approximately ϵ^2. Therefore, the effect just mentioned is small and these terms can frequently be neglected. However, when considering long-time behavior of satellites, they must be included.

In a similar way, the basic equation (7) which describes the rotational motion can be transformed. After neglecting third and higher order terms of $\bar{r}'$ and inserting the relations given by equation (23), use of $\int \bar{r}' dm = 0$ permits the integral of equation (7) to be written as

$$\int \frac{\bar{r}' \times \bar{r}}{r^3}\, dm = \int \frac{\bar{r}' \times \bar{r}_A}{r^3}\, dm = -\frac{3}{r_A^5} \int (\bar{r}' \times \bar{r}_A)(\bar{r}' \bar{r}_A)\, dm + \cdots$$
$$= -\frac{3}{r_A^3} \left[\bar{e}_x \gamma_y \gamma_z (C - B) + \bar{e}_y \gamma_z \gamma_x (A - C) + \bar{e}_z \gamma_x \gamma_y (B - A) \right] + \cdots$$

Equation (7) then becomes $\frac{d'\overline{H}_s}{dt} - \overline{H}_s \times \overline{\omega}$

$$= \frac{3\gamma m_E^4}{m^3 r_s^3}\left[\overline{e}_x \gamma_y \gamma_z (C-B) + \overline{e}_y \gamma_z \gamma_x (A-C) + \overline{e}_z \gamma_x \gamma_y (B-A)\right] + \cdots \quad (28)$$

An evaluation of the order of magnitude of the different terms reveals the influence of the right-hand term. The Coriolis term $\overline{H}_s \times \overline{\omega}$ is of the order $m_s L^2 \omega^2$ where ω is the rotational velocity of the satellite. Using equation (3) reveals that the right-hand term is of the order $m_s L^2 \Omega^2$ where $\Omega = \frac{g_0}{R}$ is the Schuler frequency. Ω is nearly the angular velocity of the radius $\overline{r}_A$ for the orbiting satellite. Consequently, the ratio of these two terms is approximately $\omega^2 : \Omega^2$. This means that fast nutational motions of the satellite with periods up to about a minute can be calculated with equation (28) by neglecting the right-hand side entirely. However, slow wobbling motions are important for certain types of satellites and the right-hand side of equation (28) must be included in these calculations.

The right-hand term of equation (28) vanishes if the satellite has spherical ellipsoids of inertia (A = B = C) or if one of the principal axes is vertical. In the latter case, one of the direction cosines vanishes. If the ellipsoid of inertia is symmetrical, then every axis perpendicular to the axis of symmetry can be considered to be a principal axis.

Using the abbreviation $K = \frac{3\gamma m_E^4}{m^3 r_s^3}$, (29)

the scaler components of the vector equation (28) are

$$\begin{aligned} A\dot{\omega}_x - (B-C)\,\omega_y\,\omega_z &= K\,\gamma_y\,\gamma_z\,(C-B) \\ B\dot{\omega}_y - (C-A)\,\omega_z\,\omega_x &= K\,\gamma_z\,\gamma_x\,(A-C) \\ C\dot{\omega}_z - (A-B)\,\omega_x\,\omega_y &= K\,\gamma_x\,\gamma_y\,(B-A) \end{aligned} \quad (30)$$

5. Small Perturbations of Regular Motions for Symmetrical Satellites

In calculating small perturbations of the regular motions (described in Chapter 3) of symmetrical satellites, it is necessary to impose restrictions not only on the ellipsoids of inertia, but also on the motion itself. Small deviations from the regular motions of a rod-shaped satellite are considered where the shape of the satellite is more general than was considered previously: the ellipsoids of inertia are assumed symmetrical ($A = B$). The small effect of the satellite orientation on the orbit is neglected so that the rotational motions of the satellite are considered when the center of mass moves on a circular orbit with constant speed.

Equations (30) can be written in more convenient form by assuming that spin about the x and y axes is small and placing no restrictions on spin about the z-axis. With this assumption the third of equations (30) shows that the changes of ω_z are small of second order. For first order theory, $\omega_z = \omega_{z_o}$ where the subscript o denotes average angular velocity about the axis of symmetry. Equations (30) are simplified by use of the shape factor $q = C/A$. For example: $q = 0$ is a rod; $q = 1$ is a satellite with spherical ellipsoid of inertia; $q = 2$ is a disc. Then the first and second of equations (30) can be written in the form

$$\begin{aligned} \dot{\omega}_x - (1-q)\,\omega_{z_o}\,\omega_y &= -(1-q)\,K\,\gamma_y\,\gamma_z \\ (1-q)\,\omega_{z_o}\,\omega_x + \dot{\omega}_y &= (1-q)\,K\,\gamma_z\,\gamma_x \end{aligned} \qquad (31)$$

A suitable method must be developed for measuring deviation from the regular position. The components of rotation ω_x, ω_y and the direction cosines γ_x, γ_y, γ_z in equations (31) are expressed in a suitable set of angles which describe the orientation of the satellite in space. The set of angles is comprised of ϕ, Ψ, δ as mentioned in Chapter 2. Choosing the x-axis as an axis of symmetry allows the angle ϕ to represent angular position about the z-axis -- a quantity which is usually of no interest for a symmetrical satellite. Therefore, ϕ can be eliminated. The angles Ψ and δ then describe the deviation of the axis of symmetry from the "regular" position being considered. These angles are either small or can be expressed by certain other small angles. In order that the results be interpreted in a descriptive way, different positions of the system of coordinates are used for the three cases. The deflection of the axis of symmetry is described by the path of the projection of an eccentric fixed point of this axis on a plane perpendicular to the "regular" position of the axis of symmetry. The projection plane itself rotates with the radius vector $\overline{r}_s$ (vertical) and with the tangent to the orbit (horizontal). Then letting α and β represent small deflections of the projected point in this plane permits the relation between α and β on one side and Ψ and δ on the other to be easily determined.

5.1. Small perturbations for case #1

A relationship between the deviation angles Ψ and δ must be established for the case of a satellite orbiting with its longitudinal axis normal to the plane of the orbit. The inertial coordinate system is given an orientation such that the orbit is situated in the ξ,ζ-plane and such that the vector of rotation $\overline{\Omega}$ of the radius vector $\overline{r}_s$ coincides with the negative direction of the η-axis (Fig. 2). If the vector $\overline{r}_s$ coincides with the ξ-axis at time t = 0, then the unit vector in the direction of the vertical has components $\overline{e}_r$ = (cos Ωt, 0, sin Ωt) in the ξ,η,ζ system. Considering Ψ and δ as small, the components of $\overline{e}_v$ in the body-fixed coordinate system x, y, z can easily be calculated by using the transformation matrix (11). The components are:

$$\begin{aligned} \gamma_x &= \cos\Omega t\cos\phi + \sin\Omega t\sin\phi = \cos(\Omega t - \phi) \\ \gamma_y &= \sin\Omega t\cos\phi - \cos\Omega t\sin\phi = \sin(\Omega t - \phi) \\ \gamma_z &= \psi\cos\Omega t + \delta\sin\Omega t \end{aligned} \tag{32}$$

On the other hand, the components of rotation can be calculated by using the kinematical Euler-equations (12). The components are:

$$\begin{aligned} \omega_x &= \dot{\psi}\sin\phi - \dot{\delta}\cos\phi \\ \omega_y &= \dot{\psi}\cos\phi + \dot{\delta}\sin\phi \\ \omega_z &= \dot{\phi} \end{aligned} \tag{33}$$

By inserting equations (32) and (33) into equations (31) and eliminating ϕ, two linear equations for Ψ and δ result. They are:

$$\begin{aligned} \ddot{\psi} + q\,\omega_{z_0}\dot{\delta} &= K(1-q)(\psi\cos^2\Omega t + \delta\sin\Omega t\cos\Omega t) \\ \ddot{\delta} - q\,\omega_{z_0}\dot{\psi} &= K(1-q)(\psi\sin\Omega t\cos\Omega t + \delta\sin^2\Omega t) \end{aligned} \tag{34}$$

The set of equations (34) can be simplified by introducing the angles α and β in accordance with figure 3. P is the projection of an eccentric point of the axis of symmetry on a plane defined by the vertical V and the horizontal tangent H_t to the orbit. With the above mentioned initial conditions (see Fig. 4), Ψ and δ in terms of α and β become

$$\begin{aligned}\psi &= \alpha \sin \Omega t + \beta \cos \Omega t \\ \delta &= -\alpha \cos \Omega t + \beta \sin \Omega t\end{aligned} \tag{35}$$

Using these equations, equations (34) become

$$\begin{aligned}&\ddot{\alpha} + \Omega(q\,\omega_{z_0} - \Omega)\alpha + (q\,\omega_{z_0} - 2\Omega)\dot{\beta} = 0 \\ &\ddot{\beta} + [\Omega(q\,\omega_{z_0} - \Omega) - (1-q)K]\beta - (q\,\omega_{z_0} - 2\Omega)\dot{\alpha} = 0\end{aligned} \tag{36}$$

The criteria for stability can now be established. The characteristic equation for this set of simultaneous equations (36) becomes

$$\begin{vmatrix} \lambda^2 + \Omega(q\,\omega_{z_0} - \Omega) & (q\,\omega_{z_0} - 2\Omega)\lambda \\ -(q\,\omega_{z_0} - 2\Omega)\lambda & \lambda^2 + \Omega(q\,\omega_{z_0} - \Omega) - (1-q)K \end{vmatrix} = 0$$

or

$$\lambda^4 + a\lambda^2 + b = 0 \tag{37}$$

where $$a = q^2\omega_{z_0}^2 - 2q\,\omega_{z_0}\Omega - \Omega^2 + 3q\Omega^2$$

and $$b = 4\Omega^2 - 5q\,\omega_{z_0}\Omega^3 + 3q^2\omega_{z_0}\Omega^3 + q^2\omega_{z_0}^2\Omega^2 - 3q\Omega^4$$

Stable solutions of (37) are possible only if the values of λ^2 are real and negative. Therefore, the criteria for stability are

$$\begin{aligned}&a > 0 \;;\; b > 0 \\ &D = a^2 - 4b > 0\end{aligned} \tag{38}$$

A vibrational type of instability (complex value for λ) exists if $b > 0$ and $D < 0$. An asymptotic type (real value for λ) exists if either $b < 0$ or $b > 0$ and $a < 0$; $D > 0$.

The regions of stability and instability for different values of $q = C/A$ and $\nu = \frac{\omega_{z_0}}{\Omega}$ are shown in figure 5. The borders of the stable region are given by $b = 0$ (border of asymptotic stability) and $D = 0$ (border of vibrational stability). Instability is not only possible for elongated bodies with $q > 1$, but also within the region $-\Omega \leq \omega_{z_0} \leq +\Omega$ for oblite satellites with $q > 1$. This disproves the generality of a conjecture mentioned by Duboschin. In summarizing the results of his calculations, he stated that the stability-properties of rod-shaped and disc-shaped satellites are direct opposite, meaning that the disc-shaped satellite is stable where the rod-shaped satellite is unstable and vice versa. As an example, consider the case I for $\omega_{z_o} = 0$ where both types are unstable. Only slightly oblated satellites with $1 < q < 1.33$ can be used in this case without spin. The situation can be improved by a spin ω_{z_o} equal to orbital frequency Ω. For this case all types of oblate symmetrical satellites are stable.

Figure 5 can be used to determine the spin ω_{z_o} necessary to stabilize symmetrical satellites. With a sufficient amount of spin, even elongated satellites with small parameter q can be stabilized. If the period of spin rotation is less than about ten minutes ($\omega_{z_o} > 8\Omega$), approximate formulae can easily be derived from the characteristic equation (37). The border of stability is given by

$$q \approx \frac{5}{\nu + 3} \quad \text{for} \quad \nu > 8$$

$$\text{and} \quad q \approx \frac{2}{\nu} \quad \text{for} \quad \nu < -8 \quad . \tag{39}$$

The roots of the characteristic equation can be approximated by

$$\begin{aligned} \lambda_1 &\approx +i q \omega_{z_o} \\ \lambda_2 &\approx -i q \omega_{z_o} \\ \lambda_3 &\approx +\Omega \\ \lambda_4 &\approx -\Omega \end{aligned}$$

These approximations indicate that the motion can be considered as a superposition of a nutational type tumbling with frequency $q\omega_{z_o}$ on a motion with orbital frequency Ω.

The transient behavior of the tumbling motion of a satellite can be determined from equation (37). For the stable cases, the motion can be characterized by the frequency-ratio $n = \frac{|\lambda|}{\Omega}$; for the unstable cases, the amount of instability can be determined by the time constant $T_c = \frac{1}{\text{real part of } \lambda}$. During the time T_c, the deflection increases by a factor $e = 2.718$. Diagrams of n and T_c for the three cases considered herein are shown in figures 6 and 7. Both diagrams are valid for non-spinning satellites ($\omega_{z_o} = 0$).

For case I, the possible modes of motion are equally valid for both angles α and β. Thus the curve in figure 6 corresponding to case I is simply denoted by "I". When two frequencies exist (for $1 \leq q \leq 1.333$), the motion is stable. One of the frequencies is smaller and the other one greater than the orbital frequency Ω which is characterized by $n = 1$.

In figure 7, the time constant T_c is calculated for orbits near the earth's surface using the Schuler-frequency as orbital frequency. The time constant is of interest only in the unstable cases. For case I, the satellite is unstable for $0 < q < 1$ and $1.333 < q < 2$. The smallest value of T_c in case I is obtained for a rod-shaped satellite ($q = 0$) where $T_c = 12.0$ minutes. An initial deflection of the axis of symmetry from the "regular" position can be increased during a single orbit by a factor of

$$e^{\frac{84.3}{12}} \approx 1100$$

Although these considerations are valid only for small values of the deflection angles α and β, the general mode of motion can be detected.

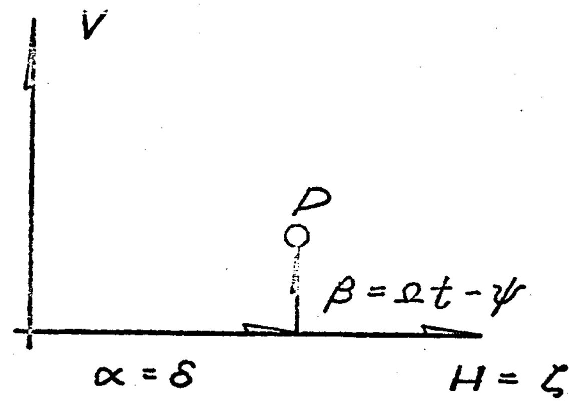

FIG 8

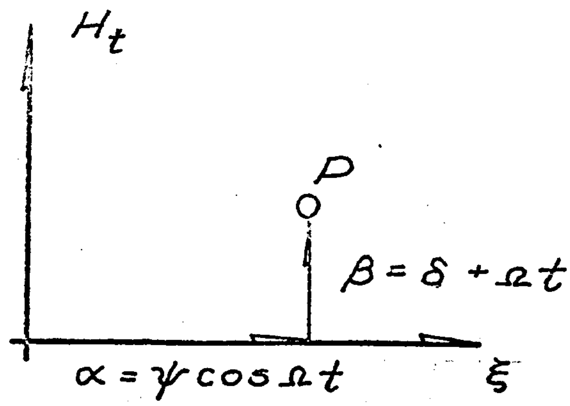

FIG 9

5.2 Small perturbations for the case II

A relationship between the deviation angles Ψ and δ must be established for the case of a satellite orbiting with its axis of symmetry tangent to the orbit. The inertial coordinate system is given an orientation such that the orbit is situated in the ξ, ζ -plane (Fig. 2). The undisturbed orientation of the satellite is given by $\Psi = \Omega t$ and $\delta = 0$. If the deviations from these values are assumed small, then the variables

$$\alpha = \delta \quad \text{AND} \quad \beta = \Omega t - \psi$$

will also be small. Once again these angles indicate the deflection of the axis of symmetry from the undisturbed position. This deflection can be represented by the projection of a point eccentric to the axis of symmetry on a plane comprised of the vertical direction and the horizontal ξ -axis (see fig. 8). If $\bar{r}_S$ coincides with the negative ξ -direction at time t = 0, then the components of $\bar{e}_v$ in the ξ, η, ζ system are

$$\bar{e}_v = (-\cos \Omega t, -\sin \Omega t, 0)$$

Since α and β are small angles, the corresponding components in the x, y, z system are

$$\bar{e}_v \approx (-\cos \phi, \sin \phi, \beta) \tag{40}$$

The components of rotation from equation (12) become

$$\omega_x = \Omega \sin \phi - \dot{\alpha} \cos \phi - \dot{\beta} \sin \phi \tag{41}$$

$$\omega_y = \Omega \cos \phi + \dot{\alpha} \sin \phi - \dot{\beta} \cos \phi$$

$$\omega_z = \Omega \alpha + \dot{\phi}$$

By inserting equations (40) and (41) into equation (31) and eliminating ϕ, two linear equations for Ψ and δ result. They are

$$\begin{aligned} &\ddot{\alpha} + \Omega^2 \alpha + \omega_{z_0} q \dot{\beta} = \omega_{z_0} q \Omega \\ &\ddot{\beta} - (1-q) K \beta - \omega_{z_0} q \dot{\alpha} = 0 \end{aligned} \tag{42}$$

The stationary solution to this set of equations is

$$\begin{aligned} &\alpha = \alpha_0 = \frac{q \omega_{z_0}}{\Omega} \\ &\beta = 0 \end{aligned} \tag{43}$$

The stationary solution (43) indicates that a horizontal deviation of the axis of symmetry from its "regular" position is necessary to compensate the gyroscopic torques caused by the orbital frequency Ω for a spinning satellite ($\omega_{z_0} \neq 0$). From equation (42), the following characteristic equation

$$\lambda^4 + \lambda^2 \left[\Omega^2 + \omega_{z_0}^2 q^2 - (1-q) K\right] - \left[\Omega^2 K (1-q)\right] = 0 \tag{44}$$

indicates instability for all elongated bodies ($q < 1$). Because these calculations are valid only for small values of α and hence α_0, equations (43) show that $q\omega_{z_0} << \Omega$. Making use of this permits simple approximations to the roots of equation (44) to be obtained. The approximate roots are

$$\left(\frac{\lambda}{\Omega}\right)_1^2 \approx -1 \quad ; \quad \left(\frac{\lambda}{\Omega}\right)_2^2 \approx 3(1-q) \tag{45}$$

These roots indicate that the motion does not depend upon the small spin ω_{z_0}, but instead upon the frequency Ω.

The stable motions of oblate bodies ($q > 1$) have not been determined. Stable motions are possible but a complete determination would require a non-linear analysis for large deflections. Such an analysis would exceed the intended extent of this investigation.

Equations (42) are decoupled if the satellite does not spin ($\omega_{z_0} = 0$). Then, by using initial conditions α_0, $\dot{\alpha}_0$, β_0, $\dot{\beta}_0$ the solutions become

$$\alpha = \alpha_0 \cos \Omega t + \frac{\dot{\alpha}_0}{\Omega} \sin \Omega t$$

$$\beta = \beta_0 \cosh \sqrt{(1-q)K}\,t + \frac{\dot{\beta}_0}{\sqrt{(1-q)K}} \sinh \sqrt{(1-q)K}\,t \qquad \text{FOR } q<1$$

$$\beta = \beta_0 + \dot{\beta}_0 t \qquad \text{FOR } q = 1 \tag{46}$$

$$\beta = \beta_0 \cos \sqrt{(q-1)K}\,t + \frac{\dot{\beta}_0}{\sqrt{(q-1)K}} \sin \sqrt{(q-1)K}\,t \qquad \text{FOR } q>1$$

The body can make undamped vibrations with the orbital frequency Ω in the horizontal direction whereas stable undamped vibrations are possible only for oblated bodies ($q \geq 1$) in the vertical direction. For $q \leq 1$, the slightest perturbation in the vertical direction would give an aperiodic drift of the axis of symmetry away from the position of equilibrium.

The frequencies for both horizontal and vertical vibrations are depicted in figure 6. For $q = 1.333$, both frequencies are equal. The satellite then behaves like a pendulum with two degrees of freedom. The greatest possible frequency occurs for a disc-shaped satellite ($q = 2$) with $n = \left|\frac{\lambda}{\Omega}\right| = 1.732$. This corresponds to a period of vibration of about 58% of the orbital time. A point on the axis of symmetry describes a Lissajous-pattern with 1.732 vertical vibrations during one horizontal vibration (for the case $q = 2$).

The degree of instability for $q < 1$ can be evaluated by means of the time constants T_C plotted in figure 7. For case II, only one branch (II; β) exists. The largest degree of instability is for a rod-shaped satellite ($q = 0$) with $T_C = 7.8$ minutes for an orbit near the earth's surface. An initial misalignment can increase by $e^{\frac{84.3}{7.8}} \approx 50{,}000$ during a single orbit.

5.3. Small perturbations for case III

A relationship between the deviation angles Ψ and δ must be established for the case of a satellite orbiting with its longitudinal axis always pointing to the center of the earth. The inertial coordinate system is given an orientation such that the orbit is situated in the η, ζ-plane and such that the axis of symmetry initially coincides with the η-axis. Then the unit vector in the direction of the vertical has components $\bar{e}_V = (0, \cos \Omega t, \sin \Omega t)$ in the ξ, η, ζ-system. The undisturbed position is given by $\Psi = 0$ and $\delta = -\Omega t$, as shown in figure 2. In this case, the angles α and β become

$$\begin{aligned} \alpha &= \psi \cos \Omega t \\ \beta &= \delta + \Omega t \end{aligned} \tag{47}$$

These angles determine, in a plane comprised of the ξ-axis and the tangent to the orbit H_Z (see Fig. 9), the small deflection of the axis of symmetry from its regular position. Then the x,y,z components of $\bar{e}_V$ are

$$\begin{aligned} \gamma_x &= \alpha \cos \phi + \beta \sin \phi \\ \gamma_y &= -\alpha \sin \phi + \beta \cos \phi \\ \gamma_z &= -1 \end{aligned}$$

The components of rotation from equation (12) become

$$\begin{aligned} \omega_x &= \Omega \cos \phi + \dot{\psi} \cos \Omega t \sin \phi - \dot{\beta} \cos \phi \\ \omega_y &= -\Omega \sin \phi + \dot{\psi} \cos \Omega t \cos \phi + \dot{\beta} \sin \phi \\ \omega_z &= -\dot{\psi} \sin \Omega t + \dot{\phi} \end{aligned}$$

Using these values and equations (47), equations (31) are transformed into

$$\begin{aligned} &\ddot{\alpha} + [\Omega^2 + (1-q)K]\alpha + \omega_{z_0} q \dot{\beta} = \omega_{z_0} q \Omega \\ &\ddot{\beta} + (1-q)K\beta - \omega_{z_0} q \dot{\alpha} - (\omega_{z_0} q \tan \Omega t)\alpha = 0 \end{aligned} \tag{48}$$

Because of the time dependent coefficient in the second equation, a steady state solution is non-existent for spinning bodies ($\omega_{z_0} \neq 0$). Therefore, only non-spinning satellites are considered ($\omega_{z_0} = 0$) in this report. Once again the two equations (48) become decoupled. Using the abbreviations

$$\omega_\alpha^2 = \Omega^2 + (1-q)K = -\mu_\alpha^2$$
$$\omega_\beta^2 = (1-q)K = -\mu_\beta^2$$

the solutions to equations (48) become

$$\alpha = \alpha_0 \cos \omega_\alpha t + \frac{\dot{\alpha}_0}{\omega_\alpha} \sin \omega_\alpha t \qquad \text{FOR } q < 1 + \frac{\Omega^2}{K} \approx 1.33$$
$$\alpha = \alpha_0 + \dot{\alpha}_0 t \qquad \text{FOR } q = 1 + \frac{\Omega^2}{K}$$
$$\alpha = \alpha_0 \cosh \mu_\alpha t + \frac{\dot{\alpha}_0}{\mu_\alpha} \sinh \mu_\alpha t \qquad \text{FOR } q > 1 + \frac{\Omega^2}{K}$$
$$\beta = \beta_0 \cos \omega_\beta t + \frac{\dot{\beta}_0}{\omega_\beta} \sin \omega_\beta t \qquad \text{FOR } q < 1$$
$$\beta = \beta_0 + \dot{\beta}_0 t \qquad \text{FOR } q = 1$$
$$\beta = \beta_0 \cosh \mu_\beta t + \frac{\dot{\beta}_0}{\mu_\beta} \sinh \mu_\beta t \qquad \text{FOR } q > 1$$

The solutions to equations (48) indicate stability for all types of elongated satellites ($q < 1$) and instability for oblated ones ($q > 1$). The corresponding frequencies are depicted in figure 6, curves III_α and III_β. Once again a point on the axis of symmetry describes a Lissajous-pattern. The highest possible frequency is just 2Ω meaning that a rod-shaped satellite makes two full vibrations in the plane perpendicular to the orbit during one orbit.

The degree of instability can be determined from the time constants T_c plotted for the unstable cases in figure 7, curves III_α and III_β. The smallest T_c occurs for a disc-shaped satellite ($q = 2$) and has values of $T_\alpha = 9.5$ minutes and $T_\beta = 7.8$ minutes. The initial misalignment of the axis of symmetry can therefore be increased by a factor of approximately 50,000.

It is remarkable that the periods of vibration and the time constants are nearly independent of length of the satellite. The influence of the size of the satellite is of the order $\left(\frac{L}{R}\right)^2$. For approximate calculation, this effect can be fully neglected. This shows that, even for very small satellites, the surrounding gravity field cannot be considered homogeneous when long periodic rotational motions are of interest.

6. Results

The direction of the gravity force acting on a finite body in a radial gravity field does not generally pass through the center of mass of that body. This results in (1) a torque about the center of mass and (2) a non-central component of the force. These conditions produce a coupling between the orbital and rotational motions; hence, the equations of motion must be treated simultaneously. Both equations cannot be treated separately as is usually done in the classical problems of celestial mechanics. The equations of motion are derived in detail in this report using the restrictions mentioned in the first chapter. These equations, together with the kinematical equations necessary to describe the orientation of the satellite in space, constitute a system of differential equations of twelfth order.

Three partial solutions for the exact equations, previously mentioned by Duboschin, are treated more in detail. These partial solutions show the effects of orientation of the axis of symmetry upon the period of circular orbits. In addition to the well-known corrections for the relation between masses of the earth and satellite and for the distance of the orbit from the earth's surface, a third correction is necessary to consider the orientation of a satellite in orbit. In the case of a rod-shaped satellite for instance, the orbiting time is less if the axis of the rod always points toward the center of the earth than it is if the axis is always tangential to the orbit. The center of masses in both cases will have identical circular orbits.

A series development of the basic equations (4) and (7) in terms of the ratio $\frac{L}{R}$ (length of the satellite to radius of the earth) shows that the influence of satellite orientation upon the orbital motion is of second order and hence small. This causes a non Keplerian motion with a slightly altered period of revolution and also a certain deformation of the orbit. Orbits not situated in a plane are possible. The development in power series shows furthermore that the gravity gradient torques must be taken into account if slow tumbling motions are of interest. Tumbling is considered slow when the period is of the order of magnitude of the orbiting period. For nutational motion of spinning satellites, however, calculations by means of the classical Euler-equations (which neglect gravity-gradient torques) are usually sufficiently accurate.

An analysis of the disturbed motions of symmetrical satellites shows the possible types of motion and their stability. Symmetrical satellites with an orientation of the axis of symmetry perpendicular to the plane of the orbit can be stabilized by introducing a spin around the axis of symmetry. The necessary value of spin depends upon the ratio of the principal moments of inertia. Satellites without spin are in this case stable only for a small range $1 < q < 1.33$, meaning only for slightly oblated satellites. In contradiction to a statement by Duboschin, the rod-shaped and disc-shaped satellites are unstable in this particular case. In case of a symmetrical satellite with the axis of symmetry tangential to the orbit or deflecting only slightly from this position, stable motions are possible for oblate bodies ($q > 1$). If, however, the axis of symmetry points to the center of the earth, only elongated bodies can give stable steady-state motions in a circular orbit.

The frequencies of possible vibrations are plotted versus shape-factor $q = C/A$ in fig. 6 for the stable configurations; time constants T_C are plotted versus shape-factor q in fig. 7 for the unstable configurations. The time constant is a suitable measure of the degree of instability: within a period of time T_c, an initial misalignment of the axis of symmetry is increased by a factor $e = 2.718$. For the more severe cases, the time constant is so small that an initial error may be increased by a factor of approximately 50,000 in a single orbit. For these cases, however, the linear analysis performed in this investigation is insufficient for evaluation of the motion in question over a longer time.

A remarkable result is that the transient behavior of disturbed satellites, characterized by the frequency and time constant curves of figs. 6 and 7 respectively, is nearly independent of the size of the satellite. This indicates that even for very small satellites of perhaps no more than an inch in length, the gravity-field cannot be considered homogeneous. Neglecting the radial properties of the gravity field would result in a loss of the effects mentioned here.

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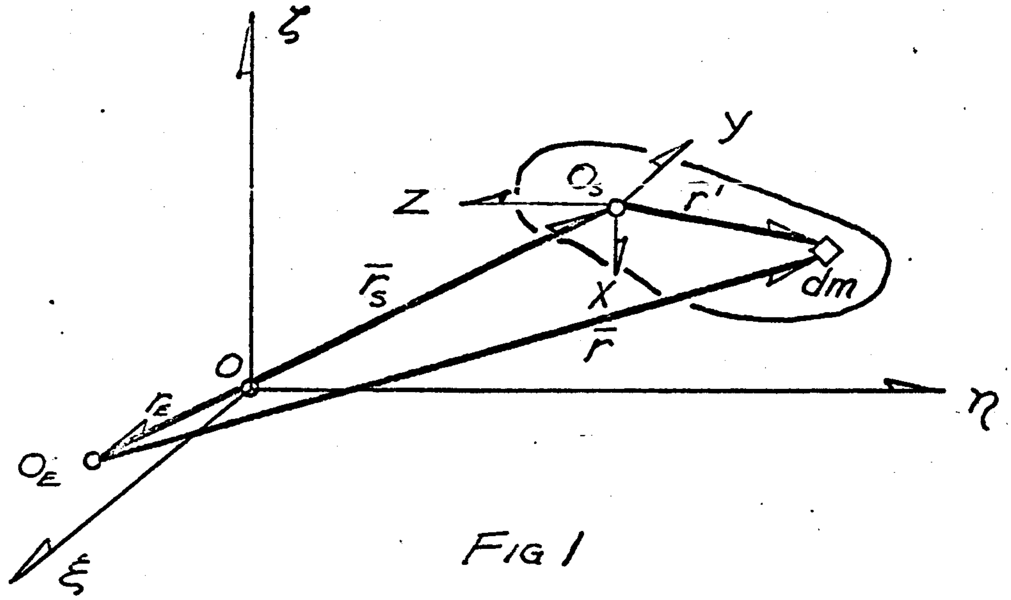


FIG 1

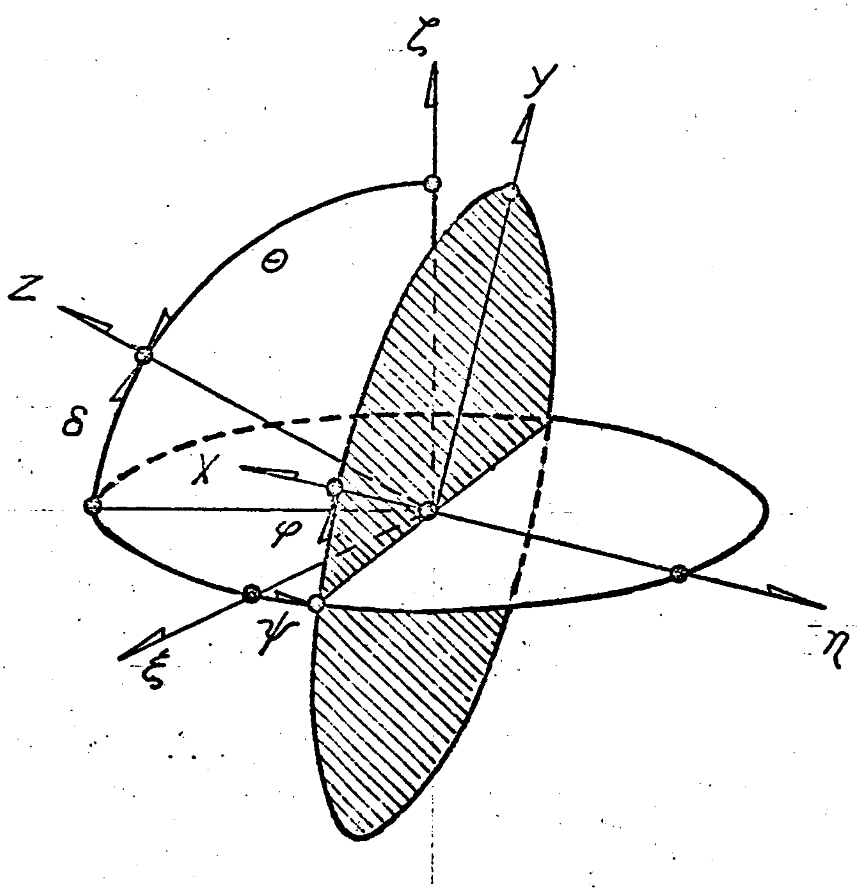


FIG 2

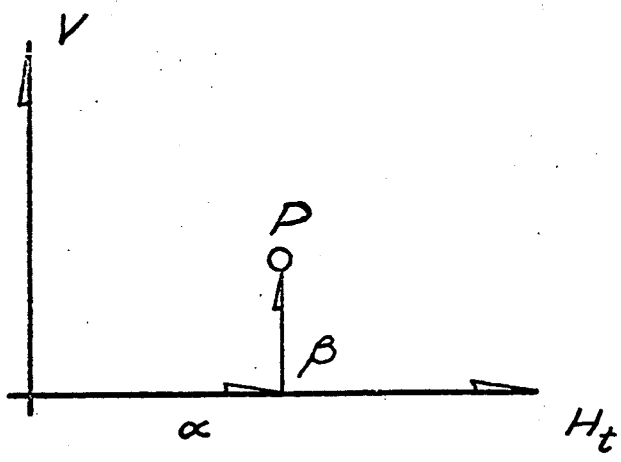


FIG 3

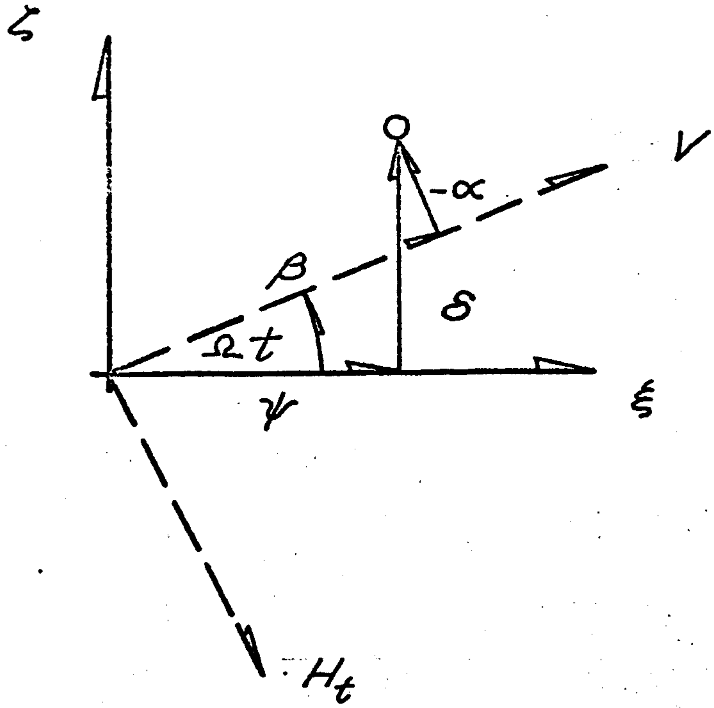
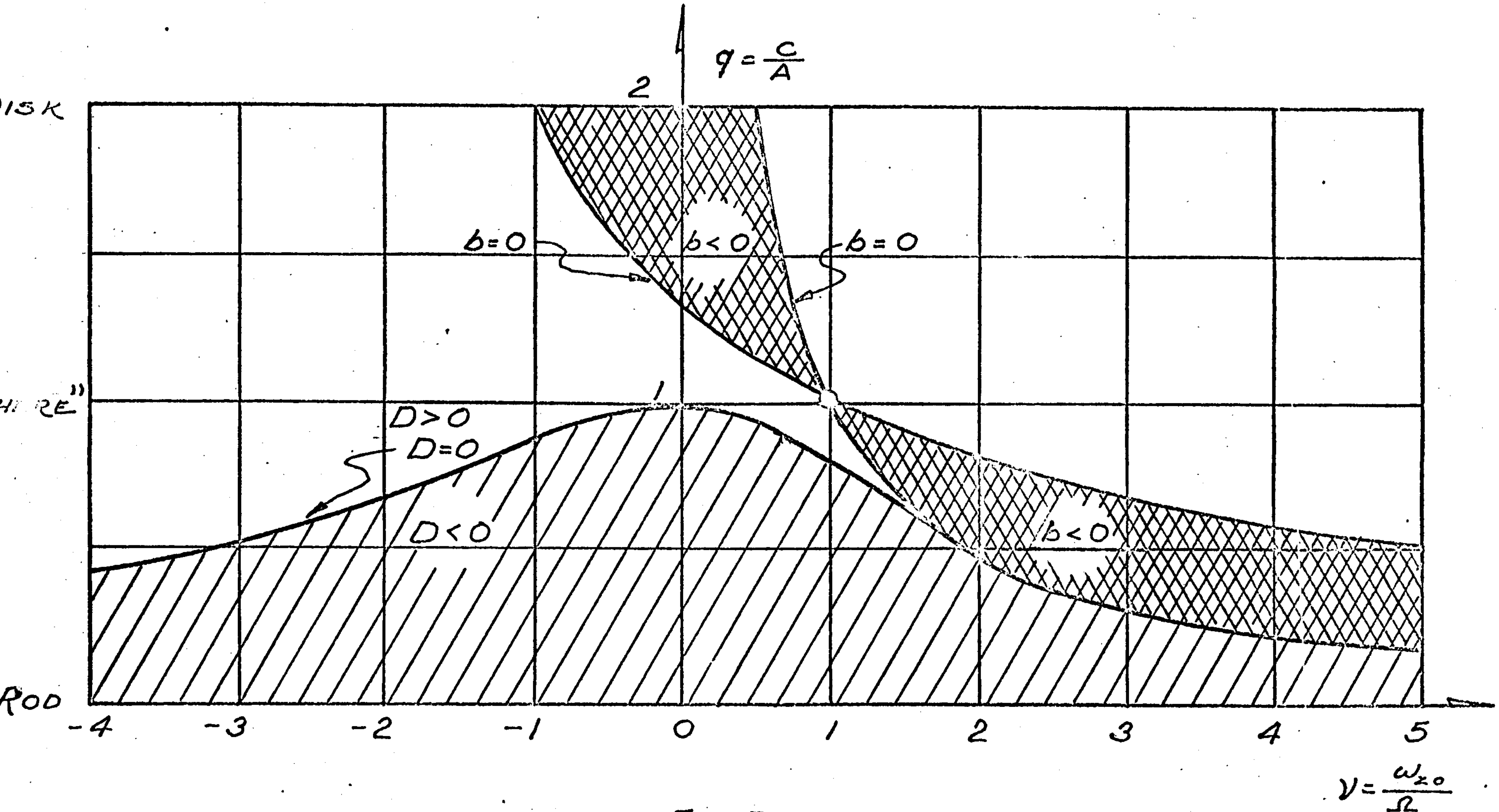
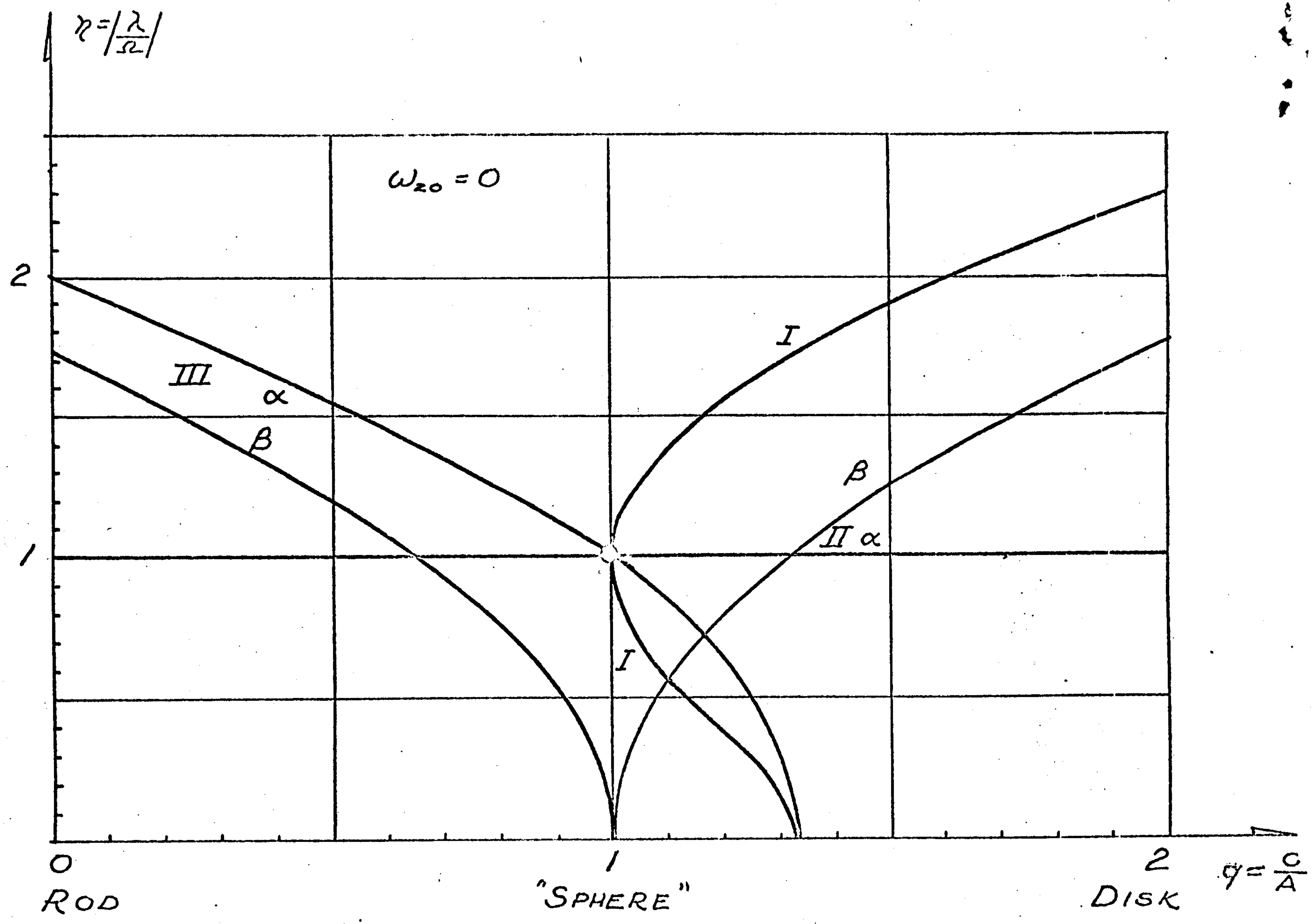


FIG 4



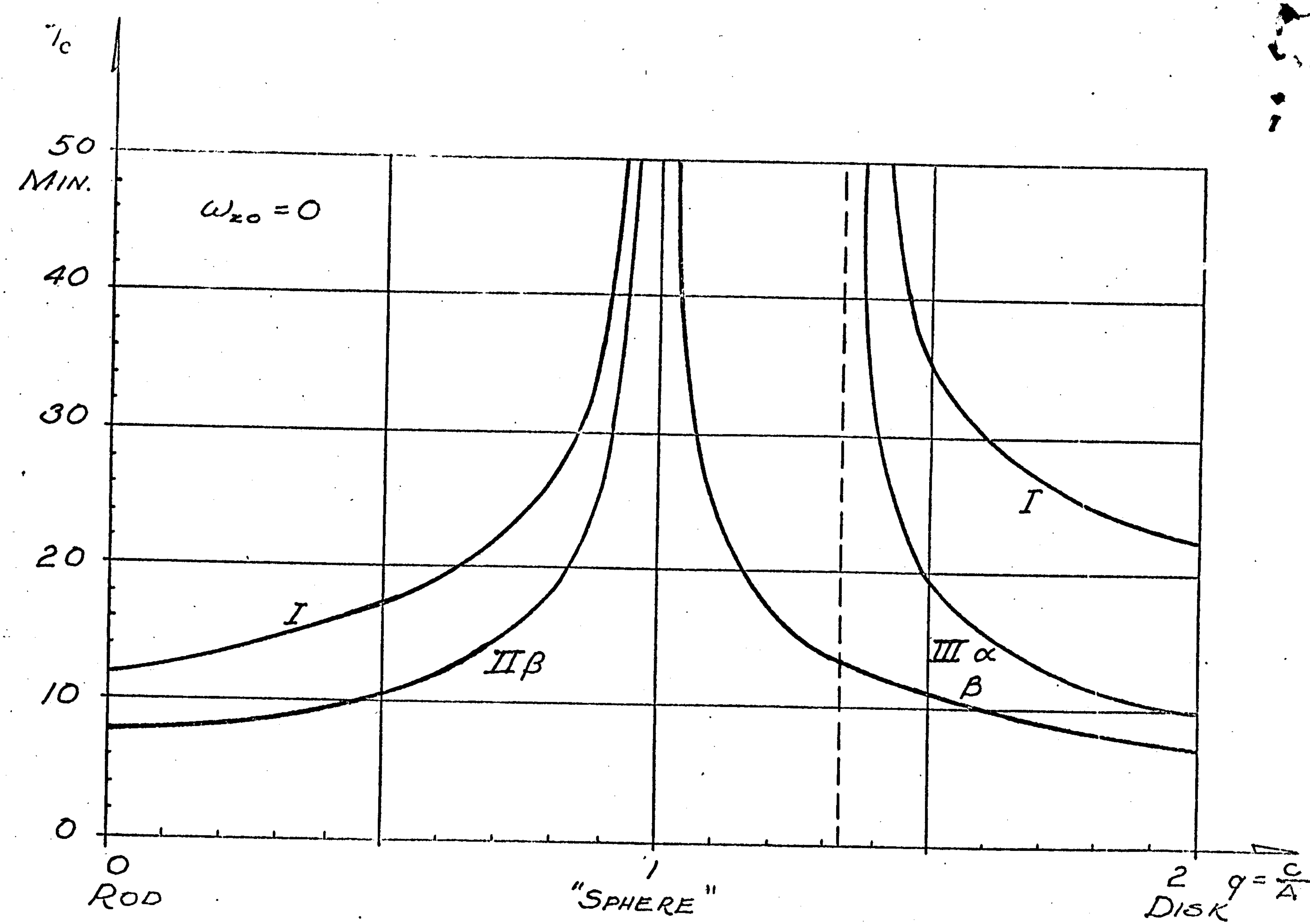
$q = \frac{C}{A}$
2
1
b=0
b<0
b=0
D>0
D=0
D<0
b<0
-4
-3
-2
-1
0
1
2
3
4
5
$\nu = \frac{\omega_{z0}}{\Omega}$

FIG 5



$\eta = \left|\frac{\lambda}{\Omega}\right|$
$\omega_{20} = 0$
2
1
III
α
β
I
β
II α
I
0
1
2
$q = \frac{C}{A}$
ROD
"SPHERE"
DISK

FIG 6



$^{\nu}/_c$
50
MIN.
40
30
20
10
0
$\omega_{z0} = 0$
I
$II\beta$
I
$III\ \alpha$
β
0
ROD
1
"SPHERE"
2
DISK
$q = \frac{c}{A}$

FIG 7